Every triangle in this diagram is similar to the Pythagorean Triangle with sides $(5, 12, 13)$.

Area of square $DGHJ = (156)^2 = 24336$.

Area of triangle $YHG = \frac{1}{2}(65)(156) = 5070$.

Area of trapezoid $EXGC = (60)^2 - \frac{1}{2}(25)(60) = 2850$.

So the desired ratio is $\dfrac{5070 + 2850}{24336} = \dfrac{55}{169}$.

Also solved by Jeremiah Bartz and Nicholas Newman, University of North Dakota and Troy University respectively, Grand Forks, ND and Troy, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania; Sachit Misra, Nelhi, India; Boris Rays, Brooklyn, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5447:** *Proposed by Iuliana Trască, Scornicesti, Romanai*

Show that if x, y, and z is each a positive real number, then

$$\frac{x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3}{x^2 \cdot y^2 \cdot z^2} \geq \frac{x^3 + y^3 + z^3 + 3x \cdot y \cdot z}{2}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The stated inequality is equivalent to

$$2x^6z^3 + 2y^6x^3 + 2z^6y^3 \geq x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5 + 3x^3y^3z^3. \qquad (1)$$

By the AM-GM inequality,

$$\sum_{cycl} x^6z^3 = \sum_{cycl}\left(\frac{2}{3}x^6z^3 + \frac{1}{3}y^6x^3\right) \geq \sum_{cycl}\left(x^{\frac{2}{3}\cdot 6}z^{\frac{2}{3}\cdot 6}y^{\frac{1}{3}\cdot 6}x^{\frac{1}{3}\cdot 3}\right) = \sum_{cycl} x^5y^2z^2,$$

$$\sum_{cycl} x^6z^3 \geq 3x^3y^3z^3$$

Statement (1) follows by adding these two inequalities.

Solution 2 by Arkady Alt, San Jose, CA

Note that,

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2\ y^2\ z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2} \iff 2\left(x^6z^3 + y^6x^3 + z^6y^3\right)$$

$$\geq \quad x^5\ y^2\ z^2 + x^2\ y^5\ z^2 + x^2\ y^2\ z^5 + 3x^3\ y^3\ z^3.$$

By AM-GM Inequality,

$$x^6 z^3 + y^6 x^3 + z^6 y^3 \geq 3\sqrt[3]{x^6 z^3 \cdot y^6 x^3 \cdot z^6 y^3} = 3\sqrt[3]{x^9 y^9 z^9} = 3x^3\ y^3\ z^3.$$

And again by AM-GM Inequality.

$$2x^6 z^3 + y^6 x^3 \geq 3\sqrt[3]{\left(x^6 z^3\right)^2 y^6 x^3} = 3\sqrt[3]{x^{15} y^6 z^6} = 3x^5 y^2 z^2,$$

and therefore,

$$3\sum_{cyc} x^6 z^3 = \sum_{cyc}\left(2x^6 z^3 + y^6 x^3\right) \geq \sum_{cyc} 3x^5 y^2 z^2 \iff \sum_{cyc} x^6 z^3 \geq \sum_{cyc} x^5\ y^2\ z^2.$$

Thus, $2\sum\limits_{cyc} x^6 z^3 = \sum\limits_{cyc} x^6 z^3 + \sum\limits_{cyc} x^6 z^3 \geq \sum\limits_{cyc} x^5\ y^2\ z^2 + 3x^3\ y^3\ z^3.$

Solution 3 by Moti Levy, Rehovot, Israel

By Muirhead inequality ($(6,3,0)$ majorizes $(5,2,2)$),

$$\sum_{sym} x^6 x^3 z^0 \geq \sum_{sym} x^5 y^2 z^2,$$

or explicitly,

$$\left(x^6 z^3 + y^6 x^3 + z^6 y^3\right) + \left(x^6 y^3 + v^6 z^3 + z^6 x^3\right) \geq 2\left(x^5 y^2 z^2 + x^2 y^5 z^2 + x^2 y^2 z^5\right). \qquad (1)$$

Again, by Muirhead inequality ($(5,2,2)$ majorizes $(3,3,3)$),

$$\sum_{sym} x^5 y^2 z^2 \geq \sum_{sym} x^3 y^3 z^3$$

or explicitly,

$$x^5 y^2 z^2 + x^2 y^5 z^2 + x^2 y^2 z^5 \geq 3x^3 y^3 z^3. \qquad (2)$$

Given three positive numbers a, b, c. We can always assign their values to x, y and z respectively, such that $x^6 z^3 + y^6 x^3 + z^6 y^3 \geq x^6 y^3 + y^6 z^3 + z^6 x^3$. Hence, without loss of generality, we can assume that

$$x^6 z^3 + y^6 x^3 + z^6 y^3 \geq x^6 y^3 + y^6 z^3 + z^6 x^3, \qquad (3)$$

then by (1), (2) and (3)

$$\begin{aligned}
2\left(x^6 z^3 + y^6 x^3 + z^6 y^3\right) &\geq \left(x^6 z^3 + y^6 x^3 + z^6 y^3\right) + \left(x^6 y^3 + v^6 z^3 + z^6 x^3\right) \\
&\geq 2\left(x^5 y^2 z^2 + x^2 y^5 z^2 + x^2 y^2 z^5\right) \\
&\geq x^5 y^2 z^2 + x^2 y^5 z^2 + x^2 y^2 z^5 + 3x^3 y^3 z^3.
\end{aligned}$$

which is equivalent to

$$\frac{x^6 z^3 + y^6 x^3 + z^6 y^3}{x^2 y^2 z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2}.$$